On a different approach to the bi-Hamiltonian structure of higher-order water-wave equations

1990 J. Phys. A: Math. Gen. 23 L591

(http://iopscience.iop.org/0305-4470/23/12/002)

J. Phys. A: Math. Gen. 23 (1990) L591-L597. Printed in the UK

LETTER TO THE EDITOR

On a different approach to the bi-Hamiltonian structure of higher-order water-wave equations

Sasanka Purkait and A Roy Chowdhury

High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta 700 032, India

Received 25 August 1989

Abstract. Conservation laws and the bi-Hamiltonian structure of higher-order water-wave equations are obtained by a method different from the usual approach of symmetry analysis. The method is based on the technique of Fourier analysis and small amplitude expansion. Some comments are made about the symmetries of the equation. The recursion operator Λ is then constructed from the two symplectic operators and it is explicitly verified that it is both a strong operator and a hereditary one.

Properties of nonlinear integrable systems can be analysed in much detail if the bi-Hamiltonian structure and associated conservation laws are known [1]. Usually such properties can be deduced if the Lax pair and Lie-Backlund symmetries are known [2]. Many nonlinear equations have already been studied from this point of view. Here we have shown that it is possible to develop a different approach to extract these properties without the use of the Lax pair. The equation under consideration is the higher-order water-wave equation of which the Boussinesq equation is a special case [3]. Our approach is that of small amplitude expansion and Fourier expansion. The technique of Fourier expansion has already been utilised by Chen *et al* [4] and others in obtaining the alternative Lax pair for some equations such as KdV, MKdV etc [5].

The equation under consideration can be written as

$$\begin{aligned} &u_t+(uv)_x+\tfrac{1}{3}v_{xxx}=0 \\ &v_t+u_x+vv_x=0. \end{aligned} \tag{1}$$

If we denote the vector $\binom{u}{v}$ as P then (1) can be written as

$$P_t=K(u,v) \tag{2}$$

where K is also a vector,

$$\begin{aligned} &K_1=-(uv)_x-\tfrac{1}{3}v_{xxx} \\ &K_2=-u_x-vv_x. \end{aligned} \tag{3}$$

Now let us linearise (2) by setting $u\to u+\varepsilon\sigma'$, $v\to v+\varepsilon\eta'$; then if the vector $\binom{\sigma'}{\eta'}$ is denoted by R, we get

$$R_t=K'(u,v)R \tag{4}$$

where the matrix $K'(u,v)$ is obtained from the Frechet derivative of $K(u,v)$:

$$K'(u,v)=\begin{pmatrix} -\partial v & -\tfrac{1}{3}\partial^3-\partial u \\ -\partial & -\partial v \end{pmatrix}. \tag{5}$$

We assume that $(u, v) \in M = C_l^{(\alpha)}(R^1, R^2)$ a smooth space of an l-periodic function $x \in R^1$, $t \in R^1$. Our motivation is to study the existence of an infinite heirarchy of conservation laws on M. For this purpose we study the asymptotic solution of the adjoint equation of (4), denoted as $\varphi \in T^*(M)$; let '*' denote the conjugate with respect to the standard bilinear form (,) on $T^*(M) \times T(M)$. Now $K'^*: T^*(M)$; $K'^*: T^*(M) \to T^*(M)$ has the form

$$K'^* = \begin{pmatrix} v\partial & \partial \\ \frac{1}{3}\partial^3 + u\partial & v\partial \end{pmatrix} \tag{6}$$

and we have, for the equation adjoint to (4),

$$\varphi_t + K'^*\varphi = 0. \tag{7}$$

The solution vector φ has the form

$$\varphi = \begin{pmatrix} 1 \\ b(k) \end{pmatrix} \exp\left(kx - \frac{k^2}{\sqrt{3}}t\right) + \int_{x_0}^{x} \sigma \, dx \tag{8}$$

where k is a complex parameter, $x_0 \in R^1$ is a chosen point and

$$\begin{aligned} \sigma(x, k) &= \sum_{j \in z_+} \sigma_j[u, v]k^{-j} \\ b(k) &= \sum_{j=-1, z_+} b_j[u, v]k^{-j}. \end{aligned} \tag{9}$$

Substituting (8) and (9) into (7) and equating various powers of k^{-j} we get

$$-\frac{1}{\sqrt{3}}\delta_{j,-2} + \int_{x_0}^{x} \sigma_{j,t} \, dx = -v\delta_{j,-1} - v\sigma_j = b_{jx} - b_{j+1} - \sum_l b_{j-l}\sigma_l \tag{10}$$

$$\begin{aligned} b_{jt} + \sum_l b_{j-l} \int_{x_0}^{x} \sigma_{lt} \, dx - \frac{1}{\sqrt{3}} b_{j+2} \\ = -\tfrac{1}{3}\sigma_{jxx} - \sigma_{j+1,x} - \sum_l \sigma_{j-l}\sigma_{lx} - \tfrac{1}{3}\delta_{j,-3} - \sigma_{j+2} - \sum \sigma_{j+1-l}\sigma_l \\ -\tfrac{1}{3}\sum_{l,s} \sigma_{j-l}\sigma_{l-s}\sigma_s - u\delta_{j,-1} - u\sigma_j - vb_{jk} - vb_{j+1} - v\sum_l b_{j-l}\sigma_l. \end{aligned} \tag{11}$$

If we solve (10) and (11) recursively we then get

$$\begin{aligned} \sigma_0 &= -\tfrac{1}{2}\sqrt{3}\, v \\ \sigma_1 &= \tfrac{1}{2}\sqrt{3}\, v_x - \tfrac{3}{2}u \\ \sigma_2 &= -\tfrac{1}{2}\sqrt{3}\, v_{xx} + \tfrac{3}{2}u_x + \tfrac{3}{4}vv_x - \tfrac{3}{4}\sqrt{3}\, uv \\ \sigma_3 &= -\tfrac{3}{4}(-\tfrac{2}{3}\sqrt{3}\, v_{xxx} + 2u_{xx} + \tfrac{3}{2}\, v_x^2 + 2vv_{xx} - 2\sqrt{3}\, uv_x - 2\sqrt{3}\, vu_x + \tfrac{3}{2}u^2 - \tfrac{3}{2}v^2v_x + \tfrac{3}{2}v^2u) \\ \sigma_4 &= -\tfrac{3}{4}(\sqrt{3}\, u_xv_x + \tfrac{1}{4}\sqrt{3}\, v_x^2 v - 9uvv_x + \tfrac{3}{2}\sqrt{3}\, uv_{xx} + \tfrac{9}{4}\sqrt{3}\, u^2v + \tfrac{1}{2}\sqrt{3}\, v^2v_{xx} \\ &\quad -\tfrac{9}{2}v^2u_x + \tfrac{3}{4}\sqrt{3}\, v^3u + \tfrac{1}{2}\sqrt{3}\, vu_{xx}) + \text{total derivative terms} \end{aligned} \tag{12}$$

as have already been analysed in many situations; the quantity $\int_{x_0}^{x_0+l} \sigma(k, u, v)\, dy$ is connected to the time derivative of the scattering data $a(k)$, and from the inverse scattering transform we know that $\partial a/\partial t = 0$, hence $\sigma(k, u, v)$ generates the whole hierarchy of conservation laws so that σ_1, σ_2, σ_3 etc are the conservation laws in the present situation. The same procedure has been followed by Chen and Lee [4]. It is quite obvious that we can generate an infinite number of conservation laws for the nonlinear system.

We now turn to the Hamiltonian analysis of this system on the manifold M. We suppose that there exists an implectic and Noetherian operator, $\mathscr{L}: T^*(M) \to T(M)$ such that

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = -\mathscr{L} \operatorname{grad} H = K(u, v) \tag{13}$$

where the functional H is necessarily a conservation law. This implectic operator satisfies

$$\mathscr{L}_t - \mathscr{L} K'^* - K' \mathscr{L} = 0 \tag{14}$$

a necessary condition of being Noetherian. Before we proceed with the actual computation of $\mathscr{L}$ we here collect some information about the Hamiltonian machinery. Let us suppose that the manifold $M = S \oplus S$. S being the Schwartz space of $C^{(\alpha)}$ functions on the real line. Let M^* denote the dualon M, with pairing defined as

$$\begin{aligned} \langle f, S \rangle &= \left\langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \right\rangle \\ &= \int (f_1(x) S_1(x) + f_2(x) S_2(x)) \, \mathrm{d}x. \end{aligned} \tag{15}$$

Then a vector field σ is symmetrical if and only if

$$\frac{\partial \sigma}{\partial t} + [K, \sigma] = 0. \tag{16}$$

A field $J(u, v): M \to M^*$ is called symplectic if it is antisymmetric with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$ and if the Jacobi identity holds for the bracket

$$\langle\!\langle X_1, X_2, X_3 \rangle\!\rangle = \langle J'[X_1] X_2, X_3 \rangle \tag{17}$$

and a field Θ is implectic if Θ is antisymmetric with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$ and if the Jacobi identity holds for

$$\langle\!\langle X_1^*, X_2^*, X_3^* \rangle\!\rangle = \langle X_1^*, \Theta'[\Theta X_2^*] X_3^* \rangle. \tag{18}$$

Lastly an operator Λ is called a recursion operator if

$$\Lambda'[K] - K'\Lambda + \Lambda K' = 0 \tag{19}$$

and it is called a hereditary operator if and only if it satisfies

$$\Lambda'[\Lambda f] g - \Lambda'[\Lambda g] f = \Lambda\{\Lambda'[f] g - \Lambda'[g] f\} \tag{20}$$

where $\Lambda'(u)[f]$ denotes the Gateaux derivative, equal to

$$\mathrm{d}/\mathrm{d}\varepsilon\{\Lambda(u + \varepsilon f)\}|_{\varepsilon=0}.$$

We derive the di-Hamiltonian structure via a new approach. Then from the two symplectic operators we construct the operator Λ and then check explicitly that the properties (19) and (20) hold in our case.

We now use the small amplitude asymptotic method to solve equation (14). We set; $u = \varepsilon u^{(1)}$, $v = \varepsilon v^{(1)}$, with $\varepsilon \to 0$ a small parameter. Then k' and K'^* have the following structures:

$$\begin{aligned} K' &= K'_0 + \varepsilon K'_1 + \cdots \\ K'^* &= K'^*_0 + \varepsilon K'^*_1 + \cdots. \end{aligned} \tag{21}$$

On the other hand our ansatz for $\mathscr{L}$ reads

$$\mathscr{L}=\mathscr{L}_0+\varepsilon\mathscr{L}_1+\varepsilon^2\mathscr{L}_2+\cdots \tag{22}$$

In the present situation

$$K_0'=\begin{pmatrix}0 & -\frac{1}{3}\partial^3\\ -\partial & 0\end{pmatrix} \qquad K_1'=\begin{pmatrix}-\partial v^{(1)} & -\partial u^{(1)}\\ 0 & -\partial v^{(1)}\end{pmatrix}$$
$$K_0'^*=\begin{pmatrix}0 & \partial\\ \frac{1}{3}\partial^3 & 0\end{pmatrix} \qquad K_1'^*=\begin{pmatrix}v^{(1)}\partial & 0\\ u^{(1)}\partial & v^{(1)}\partial\end{pmatrix}. \tag{23}$$

Substituting (22) into (14) with (23) we get:

$$\varphi_t^{(0)}+K_0'^*\varphi^{(0)}=0 \tag{24a}$$

$$-\frac{\partial}{\partial t}(\mathscr{L}_0\varphi^{(0)})=K_0'(\mathscr{L}_0\varphi^{(0)}) \tag{24b}$$

$$\frac{\partial}{\partial t}(\mathscr{L}_1\varphi^{(0)})=\mathscr{L}_0K_1'^*\varphi^{(0)}+K_1'\mathscr{L}_0\varphi^{(0)}+K_0'\mathscr{L}_1\varphi^{(0)} \tag{25}$$

etc, where we have also considered the vector $\varphi^{(0)}$ in the series form. Our chief motivation is to solve equations (24) and (25) by the approach of a Fourier expansion for $\mathscr{L}_0$, $\mathscr{L}_1$ etc.

To gain a solution for $\mathscr{L}_1$, $\mathscr{L}_0$ let us start with equation (24) and insert the Fourier expansions

$$u^{(1)}=\sum_k\frac{k}{\sqrt{3}}u_k^{(1)}\exp\left(kx-\frac{k^2}{\sqrt{3}}t\right)$$
$$v^{(1)}=\sum_k u_k^{(1)}\exp\left(kx-\frac{k^2}{\sqrt{3}}t\right) \tag{26}$$

whence equation (24a) immediately suggests

$$\varphi_1^{(0)}=\sum_k\varphi_{1k}^{(0)}\exp\left(kx-\frac{k^2}{\sqrt{3}}t\right)$$
$$\varphi_2^{(0)}=\sum_k\frac{k}{\sqrt{3}}\varphi_{1k}^{(0)}\exp\left(kx-\frac{k^2}{\sqrt{3}}t\right) \tag{27}$$

and (24b) reads

$$\xi_{1t}=-\tfrac{1}{3}\partial^3\xi_2$$
$$\xi_{2t}=-\partial\xi_1 \tag{28}$$

when we set

$$\mathscr{L}_0\varphi^{(0)}=\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}.$$

From (27) a very easy solution to (28) can be seen to be

$$\xi_1=\partial\varphi_2^{(0)} \qquad \xi_2=\partial\varphi_1^{(0)}$$

so,

$$\mathscr{L}_0 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}. \tag{29}$$

Now we refer to equation (25) which when written in full is

$$\begin{aligned} \Psi_{1t} &= \partial[u^{(1)}\varphi^{(0)}_{1x} + v^{(1)}\varphi^{(0)}_{2x}] - \partial[v^{(1)}\varphi^{(0)}_{2x} + u^{(1)}\varphi^{(0)}_{1x}] - \tfrac{1}{3}\partial^3\Psi_2 \\ \Psi_{2t} &= \partial[v^{(1)}\varphi^{(0)}_{1x}] - \partial[v^{(1)}\varphi^{(0)}_{1x}] - \partial\Psi_1 \end{aligned} \tag{30}$$

where we have set

$$\mathscr{L}_1\varphi^{(0)} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \tag{31}$$

It is interesting to observe that with the previous solution for $\varphi_1^{(0)}$, $\varphi_2^{(0)}$ equation (30) reduces to

$$\Psi_{1t} = -\tfrac{1}{3}\partial^3\Psi_2 \qquad \Psi_{2t} = -\partial\Psi_1 \tag{32}$$

The simplest solution to (31) is $\Psi_1 = \Psi_2 = 0$ and no other solution is interesting as it becomes identical to that of (ξ_1, ξ_2), so one implectic operator is obtained as

$$\mathscr{L} = -\frac{4}{9}\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}. \tag{33}$$

This operator also occurs in the case of the Boussinesq equation.

So, we have observed that the original nonlinear equation can be written as a Hamiltonian form:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \frac{4}{9}\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}\begin{pmatrix} \delta\sigma_3/\delta u \\ \delta\sigma_3/\delta v \end{pmatrix}. \tag{34}$$

Now it is very pertinent to observe that equation (24*a*) and (24*b*) can also have other types of solutions. For which it is essential that we start with a different form of $\mathscr{L}_0$. It is easy to observe that another solution for $\mathscr{L}_0$ is given by

$$\mathscr{L}_0 = -\begin{pmatrix} \tfrac{1}{3}\partial^3 & 0 \\ 0 & \partial \end{pmatrix}\frac{4\sqrt{3}}{9}. \tag{35}$$

We now use this in (30) and solving (30) we can manufacture another form of $\mathscr{L}_1$. Let us set

$$\mathscr{L}_1\begin{pmatrix} \varphi_1^{(0)} \\ \varphi_2^{(0)} \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$

so that this time the equation for (χ_1, χ_2) reads

$$\begin{aligned} \chi_{1t} + \tfrac{1}{3}\chi_{2xxx} &= -\tfrac{1}{3}(v^{(1)}\varphi^{(0)}_{1x})_{xxx} - \tfrac{1}{3}(v^{(1)}\varphi^{(0)}_{1xxx} + u^{(1)}\varphi^{(0)}_{2x})_x \\ \chi_{2t} + \chi_{1x} &= -(u^{(1)}\varphi^{(0)}_{1x} + v^{(1)}\varphi^{(0)}_{2x})_x - (v^{(1)}\varphi^{(0)}_{2x})_x. \end{aligned} \tag{36}$$

Using the Fourier expansion technique as mentioned above we can immediately obtain a solution of the form

$$\begin{aligned} \chi_1 &= -[\tfrac{1}{2}u^{(1)}_x + u^{(1)}\partial]\varphi_1^{(0)} - [\tfrac{1}{2}v^{(1)}\partial]\varphi_2^{(0)} \\ \chi_2 &= -[\tfrac{1}{2}v^{(1)}\partial + \tfrac{1}{2}v^{(1)}_x]\varphi_1^{(0)} \end{aligned} \tag{37}$$

whence we get

$$\mathscr{L}_1 = -\begin{pmatrix} \frac{1}{2}(\partial u^{(1)} + u^{(1)}\partial) & \frac{1}{2}v\partial \\ \frac{1}{2}\partial v^{(1)} & 0 \end{pmatrix} \frac{4\sqrt{3}}{9}. \tag{38}$$

The total implectic operator (the second one) turns out to be

$$M = \mathscr{L}_0 + \mathscr{L}_1 = -\frac{4\sqrt{3}}{9}\begin{pmatrix} \frac{1}{3}\partial^3 + \frac{1}{2}(\partial u + u\partial) & \frac{1}{2}v\partial \\ \frac{1}{2}\partial v & \partial \end{pmatrix}. \tag{39}$$

It is then straightforward to verify that

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = -M\begin{pmatrix} \delta\sigma_2/\delta u \\ \delta\sigma_2/\delta v \end{pmatrix} \tag{40}$$

giving the second Hamiltonian structure.

Once we have the information about the bi-Hamiltonian structure of a nonlinear equation it is quite easy to construct the symmetries associated with it. From equations (40) and (39) we can immediately write down the recursion operator for symmetries in the form

$$\Lambda = M\mathscr{L}^{-1} = \begin{pmatrix} \frac{1}{2}v & \frac{1}{3}\partial^2 + \frac{1}{2}(\partial u\partial^{-1} + u) \\ 1 & \frac{1}{2}\partial v\partial^{-1} \end{pmatrix}. \tag{41}$$

The set of equations given by (1), being translation invariant both with respect to space and time, have the starting symmetries

$$S_1 = \begin{pmatrix} u_x \\ v_x \end{pmatrix} \qquad S_2 = \begin{pmatrix} u_t \\ v_t \end{pmatrix}$$

and the general nth-order symmetry is given as

$$S_n = \Lambda^{n-1}S_1. \tag{42}$$

It is now interesting to note that the present set of equations does admit explicit (x, t)-dependent symmetry also, an example of which is

$$T_0 = \begin{pmatrix} tu_x \\ tv_x - 1 \end{pmatrix} = tS_1 - T_{-1} \tag{43}$$

with $T_{-1} = \binom{0}{-1}$. So, a heirarchy of symmetries can be written as

$$\begin{aligned} T_n &= t\Lambda^{n-1}S_1 - \Lambda^{n-1}T_{-1} \\ &= tS_n - \Lambda^{n-1}T_{-1} \end{aligned} \tag{44}$$

which depend explicitly on space and time.

Our above analysis has generated two symplectic operators for the higher-order water-wave equation. We can verify two crucial properties of the recursion operator $\Lambda = M\mathscr{L}^{-1}$ constructed out of these. Firstly, it is easy to verify that

$$\Lambda\begin{pmatrix} u_x \\ v_x \end{pmatrix} = -\begin{pmatrix} u_t \\ v_t \end{pmatrix} \tag{45}$$

and a detailed and laborious calculation immeciately shows $\Lambda'[K] = [K', \Lambda]$ along with $\Lambda'[f]g - \Lambda'[g]f = \Lambda'[\Lambda f]g - \Lambda'[\Lambda g]f$ for two arbitrary component vectors f and g. So our operator Λ is both hereditary and strong.

In our analysis we have deduced the bi-Hamiltonian structure, recursion operator and two heirarchies of symmetries for the higher-order water-wave problem using the method of Fourier expansion and small amplitude expansion. Our approach is more straightforward than that of the functional technique used by some authors for discussing the hereditary and master symmetries of nonlinear integrable PDE [6]. Usually some amount of guesswork is needed to proceed with the master symmetry approach though the results obtained are quite elegant. On the other hand in the present formalism we can construct the recursion operator first then the whole procedure becomes very simple. Lastly it is shown that explicit verification of strong and hereditary characters for the operator Λ can be carried out.

References

[1] Fuchssteiner B and Fokas A S 1981 *Physica* **4D** 47
[2] Magri F 1978 *J. Math. Phys.* **19** 1156
[3] Kaup D J 1975 *Prog. Theor. Phys.* **54** 72
[4] Chen H H, Lee Y C and Liu C S 1979 *Phys. Scr.* **20** 490
[5] Iino K, Ichikawa Y H and Wadati H 1982 *J. Phys. Soc. Japan* **51** 3724
[6] Fuchssteiner B 1982 *Prog. Theor. Phys.* **68** 1083; 1983 *Prog. Theor. Phys.* **70** 1508